SOUTHWEST RESEARCH INSTITUTE
8500 CULEBRA ROAD SAN ANTONIO, TEXAS 78206

N65 80615

Department of Structural Research
October 8, 1964

UNPUBLISHED PRELIMINARY DATA

Office of Grants and Research Contracts
Code SC
National Aeronautics and Space Administration
Washington, D.C. 20456

Subject: Task Order NASr-94(05)
Theoretical and Experimental Investigations of the Nonlinear Dynamic Response of Continuous Systems

Status Report Number 4
For Period 1 July to 30 September 1964

Gentlemen:

Enclosed is a report on the theoretical and experimental work conducted during the above period.

Very truly yours,

Thein Wah

Thein Wah
Staff Scientist

APPROVED:

Robert C. DeHart

Robert C. DeHart, Director
Department of Structural Research

TW:km
Enclosure

NORMAL MODES AND NONLINEAR VIBRATIONS

N 65 80615
Code None
CR-59777
pages 16

General

The concept of normal modes of vibration has been extended by Rosenberg[1] to nonlinear oscillations of any finite number of degrees of freedom. The writer has pointed out[2] that certain nonlinear continuous systems, characterized by the separability of the space and time variables, vibrate in normal modes in the sense defined by Rosenberg. In particular, the writer has shown that a simply-supported beam rigidly held at its ends is capable of vibrating in normal modes.

The motion of beams with axial tension is governed by the differential equation

$$EI\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

with

$$\frac{Na}{AE} = \frac{1}{2}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 dx$$

in which N is the axial force, w is the lateral deflection, a the length of the beam, A its cross-sectional area, E is Young's Modulus, I is the moment of inertia, and t is time.

In the case of a simply supported beam, a sine function in the space coordinate effectively separates the variables in (1), as is well known, and normal modes emerge very simply. As far as the writer is aware, this separability is confined to the simply supported case.

It is certainly very surprising that normal modes cannot be readily defined for nonsimply supported beams. It is difficult to accept the conclusion that the concept breaks down completely when the boundary conditions are changed. Consequently, it becomes necessary to generalize this concept so as to take into account physical systems in which variables are not separable.

While the possibility of such theoretical extension exists, it is proposed to investigate first, by an approximate numerical procedure as well as experimentally, whether such normal modes, conceived of intuitively, are stable.

Time Dependent Normal Modes

If one recalls the definition of normal modes in linear oscillations, it is evident that they can occur only under certain, quite restrictive, initial conditions. Thus, a beam will not vibrate in a normal mode unless it is started exactly in the appropriate mode, which, in turn, depends on its boundary conditions. A beam which is given an arbitrary starting shape will not vibrate in a normal mode. While the concept itself may be of considerable practical value, the point made here is that normal modes require restrictive conditions for their actual occurrence.

Returning now to the system governed by equations (1), the simply supported case (where the variables separate), is characterized by the fact that the mode shape itself is independent of time being at all times a sine function. In the case of other boundary conditions, since variables do not separate, one must suppose that the normal modes are a function of time, the mode shape altering continuously as the beam vibrates. However, if normal modes exist, they must be stable; that is, the beam must return to a certain shape periodically. If it does not, then the normal modes are unstable.

Numerical Procedure

In order to investigate analytically whether such time dependent normal modes are stable, one must appeal to physical intuition for certain basic assumptions. We suppose first that the axial tension N in (1) does not vary very rapidly so that it may be assumed to remain constant for sufficiently small intervals of time. If N is constant in the first of equations (1), it is linear and normal modes exist, together with a denumerably infinite sequence of discrete frequencies for any set of suitable boundary conditions. At the end of the time interval, one may use the second of equations (1) to calculate a new membrane tension and a new set of normal modes and natural frequencies. Physical intuition suggests that the first normal mode (that is, the normal mode corresponding to the lowest frequency) of the first time interval will merge into the first normal of the second time interval and so on for all subsequent time intervals and other normal modes. This is a fundamental assumption of the investigation.

Since normal modes depend on the initial conditions, one must specify such conditions consistently. Assuming that the beam is initially displaced with zero initial velocity, the normal modes are completely defined by the tension N.

For a constant N, one may assume that

$$w = X(x)\exp(ipt) \tag{2}$$

where X is a function of x alone.

The initial condition is specified by the tension parameter

$$\mu = \frac{Na^2}{2EI} = \frac{Aa}{4I} f^{*2}\int_0^a (X')^2 dx \tag{3}$$

in which f*, the amplitude of the displacement, is unknown to begin with, A is the cross-sectional area and prime denotes differentiation with respect to x.

For purposes of computation, it is convenient to write the relation (3) in the form

$$\mu = \lambda\frac{N^* a^2}{2EI} = \frac{Ah^2}{4I} \cdot f^2\phi \tag{4}$$

with

$$\lambda = N/N^*$$

$$f = f^*/h$$

$$\phi = a\int_0^a (X')^2 dx$$

where N* is the "buckling load" of the beam treated as a column and h is any convenient dimension of the beam cross section, such as its depth.

If we let

$$\alpha = [\mu + (u^2 + \mu^2)^{1/2}]^{1/2}$$

$$\beta = [(u^2 + \mu^2)^{1/2} - \mu]^{1/2} \tag{6}$$

where

$$u^2 = \rho\frac{a^4p^2}{EI}$$

then it may be shown by elementary procedures that for beams clamped at (x = 0, a) and for beams clamped at x = 0 and simply supported at x = a, the normal modes X(x) are given by

$$X(x) = \frac{\cosh\frac{\alpha x}{a} - \cos\frac{\beta x}{a}}{\cosh\alpha - \cos\beta} + \frac{\beta\sinh\frac{\alpha x}{a} - \alpha\sin\frac{\beta x}{a}}{\alpha\sin\beta - \beta\sinh\alpha} \tag{7}$$

The frequency equations for the two cases are, however, different.

For the clamped-clamped beam, the equation is

$$u(1 - \cosh\alpha\cos\beta) + \mu\sinh\alpha\sin\beta = 0 \tag{8}$$

and for the clamped simply-supported beam, the frequency equation is

$$\alpha\sin\beta\cosh\alpha - \beta\cos\beta\sinh\alpha = 0 \tag{9}$$

For a given μ, α, β and u must satisfy equations (6) and (8) or (9). The values of α, β and u may thus always be obtained by trial and error. With α and β known, the mode shape is completely defined by (7). It is to be noted, however, there are an infinity of the set (α, β, u) for a given μ. We are, for the present, interested only in the set corresponding to the lowest value of u.

Consider any time interval extending from $r\Delta$ to $(r + 1)\Delta$, where Δ is a suitably small quantity. At time $r\Delta$, the displacement and velocity are given by:

$$\frac{w}{h} = X_{r-1}\left[f_{r-1}\cos(p_{r-1}\Delta) + g_{r-1}\sin(p_{r-1}\Delta)\right] \tag{10}$$

$$v = \frac{\mathring{w}}{h} = X_{r-1}\,p_{r-1}\left[-f_{r-1}\sin(p_{r-1}\Delta) + g_{r-1}\cos(p_{r-1}\Delta)\right] \tag{11}$$

The tension parameter is

$$\mu_r = \frac{Ah^2}{4I}\left[f_{r-1}\cos(p_{r-1}\Delta) + g_{r-1}\sin(p_{r-1}\Delta)\right]^2 \phi_{r-1} \tag{12}$$

Knowing μ_r, one may calculate α_r, β_r, p_r and X_r. The displacement velocity and membrane tension for $(r\Delta)\leqslant t \leqslant (r+1)\Delta$ are then given by

$$\frac{w}{h} = X_r(f_r\cos p_r t + g_r\sin p_r t) \tag{13}$$

$$v = \frac{\mathring{w}}{h} = X_r p_r(-f_r\sin p_r t + g_r\cos p_r t) \tag{14}$$

$$\mu_r = \frac{Ah^2}{4I} f_r^2\,\phi_r \tag{15}$$

However, the displacement, velocity and membrane force at $t = 0$ as given by (13), (15) and (16) must be the same as those given by (10), (11) and (12) and so

$$X_{r-1}\left[f_{r-1}\cos(p_{r-1}\Delta) + g_{r-1}\sin(p_{r-1}\Delta)\right] = X_r f_r \tag{16}$$

$$X_{r-1}p_{r-1}\left[-f_{r-1}\sin(p_{r-1}\Delta) + g_{r-1}\cos(p_{r-1}\Delta)\right] = X_r p_r g_r \tag{17}$$

$$\left[f_{r-1}\cos(p_{r-1}\Delta) + g_{r-1}\sin(p_{r-1}\Delta)\right]^2\phi_{r-1} = f_r^2\phi_r \tag{18}$$

From (18), we get

$$f_r = \left[f_{r-1}\cos(p_{r-1}\Delta) + g_{r-1}\sin p_{r-1}\Delta\right]\left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2} \tag{19}$$

From (15) and (19)

$$\frac{X_{r-1}}{X_r} = \left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2}$$

Substitution in (16) yields

$$g_r = \frac{p_{r-1}}{p_r}\left[-f_{r-1}\sin(p_{r-1}\Delta) + g_{r-1}\cos(p_{r-1}\Delta)\right]\left(\frac{\phi_{r-1}}{\phi_r}\right)^{1/2} \qquad (20)$$

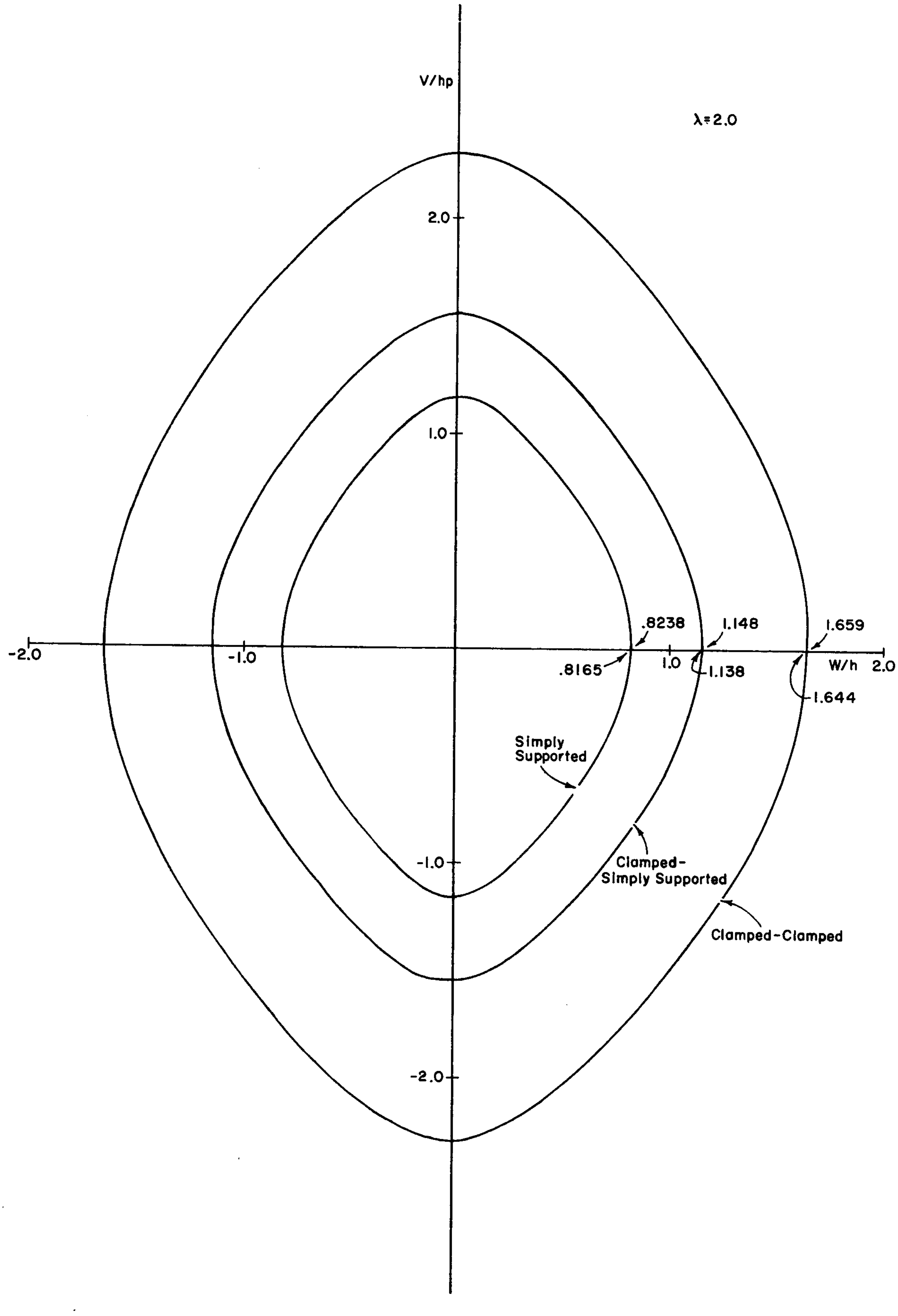
The relations (19) and (20) thus give f_r and g_r in terms of all the known quantities and enable one to proceed to the next step. It is important in this computation that the time steps be taken as small as practical as otherwise the result will indicate an unstable motion even when it is essentially stable.

Although when the beam is simply supported, an exact solution in terms of elliptic functions can be obtained, it is of interest to apply the numerical method to this case also to afford a comparison with the solutions for other boundary conditions.

Figure 1 shows a plot of the center deflection ratio (w/h) against the velocity/frequency ratio (v/h)/p, the so called phase plane. If the motion considered is stable and periodic, a closed curve should result.

The parameters chosen in the computations were those of the steel specimens to be used in the experimental work in progress and are as follows:

length = 30 in. depth = 0.25 in., width = 0.30 in.



V/hp
λ=2.0
2.0
1.0
-1.0
-2.0
-2.0
-1.0
1.0
W/h
2.0
.8238
.8165
1.148
1.138
1.659
1.644
Simply Supported
Clamped-Simply Supported
Clamped-Clamped

FIGURE 1

The resulting nondimensional parameters are

$$A_1 = (EI/a^4\rho)^{1/2} = 19.456$$

$$A_2 = (Ah^2/4I) = 3$$

The parameter λ was taken as 2.

In Figure 1 it may be noted that all the curves are essentially closed, although not exactly so. Extended computations, with various values of the time interval show, however, that the gap is a function of the time interval and decreases as the interval decreases. In the computations the quantity $(p\Delta)$ was taken as .005 and thus over 1000 intervals were required to complete 2π radians. A further decrease in the time interval seemed unwarranted, especially as it has been established that any slight gap at closure was an approximation error.

We conclude therefore that the motion is stable and periodic for all boundary conditions, and that is meaningful to speak of a time-dependent normal mode.

Period of "Fundamental" Mode

The nonlinear period T of vibration for the "fundamental" mode may be written in the following form

$$T = 4K/p \tag{21}$$

$$p = a^2 A_1 \left(1 + \nu A_2 \frac{f_o^2}{h^2}\right)^{1/2} \tag{22}$$

$$m = \frac{1}{2} \nu A_2 \frac{f_o^2}{h^2} \Big/ \left(1 + \nu A_2 \frac{f_o^2}{h^2}\right) \tag{23}$$

in which K is the complete elliptic integral of the first kind, p is the frequency, f_o is the initial displacement and m is the parameter of the Jacobian elliptic functions. The parameters α and ν have the following values for the various boundary conditions

(a) Simply supported

$$\alpha = \pi, \nu = 1$$

(b) Clamped-clamped

$$\alpha = 4.73, \nu = .3077$$

(c) Clamped-simply supported

$$\alpha = 3.927, \nu = 1.6697$$

Formulas (21), (22) and (23) are theoretically exact only for the simply supported case. For the other two cases they are approximate, having been arrived at by applying a Galerkin approximation to the differential equation[3].

Table 1 gives a comparison of the fundamental period as arrived at by formula (21) and by the numerical procedure outlined in this report. It will be noted that the agreement is close for the simply supported case only. For the other boundary conditions, formula (21) underestimates the period as might be expected.

TABLE I

FUNDAMENTAL PERIOD T (IN SECS)

	SS	CC	CS
Formula (21)	.02088	.00860	.00878
Num. Proc.	.02080	.00934	.01348

SS = Simply supported

CC = clamped-clamped

CS = clamped-simply supported

STATUS OF EXPERIMENTAL WORK

A facility for experimentally determining the vibration characteristic of beams with various boundary conditions by the Moiré method has been assembled, and preliminary static calibration tests have been completed.

Comparative grid tests were made with lines of 0.05 inch, 0.1 inch and 0.125 inch widths to determine the preferable grid system for optimum definition of the Moiré patterns. Results favored the 0.1-inch line and a grid with an overall size of 20 inches by 60 inches, mounted on 15-gage steel plate, will be used to conduct future tests.

Tests thus far were made with a high strength steel beam of 22 inches cantilevered length and 0.250 inch by 0.200 inch cross section. Two beams of different configurations have been designed--one with both ends clamped, the other with both ends pinned (Fig. 2). Each beam will be approximately 30 inches long, with a 0.250 inch of 0.300 inch cross section; fabricated from high strength steel plate, and having a .25-inch reflective surface machined to a No. 2 finish. Experiments using these beams will commence after all tests on the cantilevered beam have been completed.

Two preliminary dynamic tests were conducted on the cantilevered beam to determine the effectiveness of camera equipment assembled.

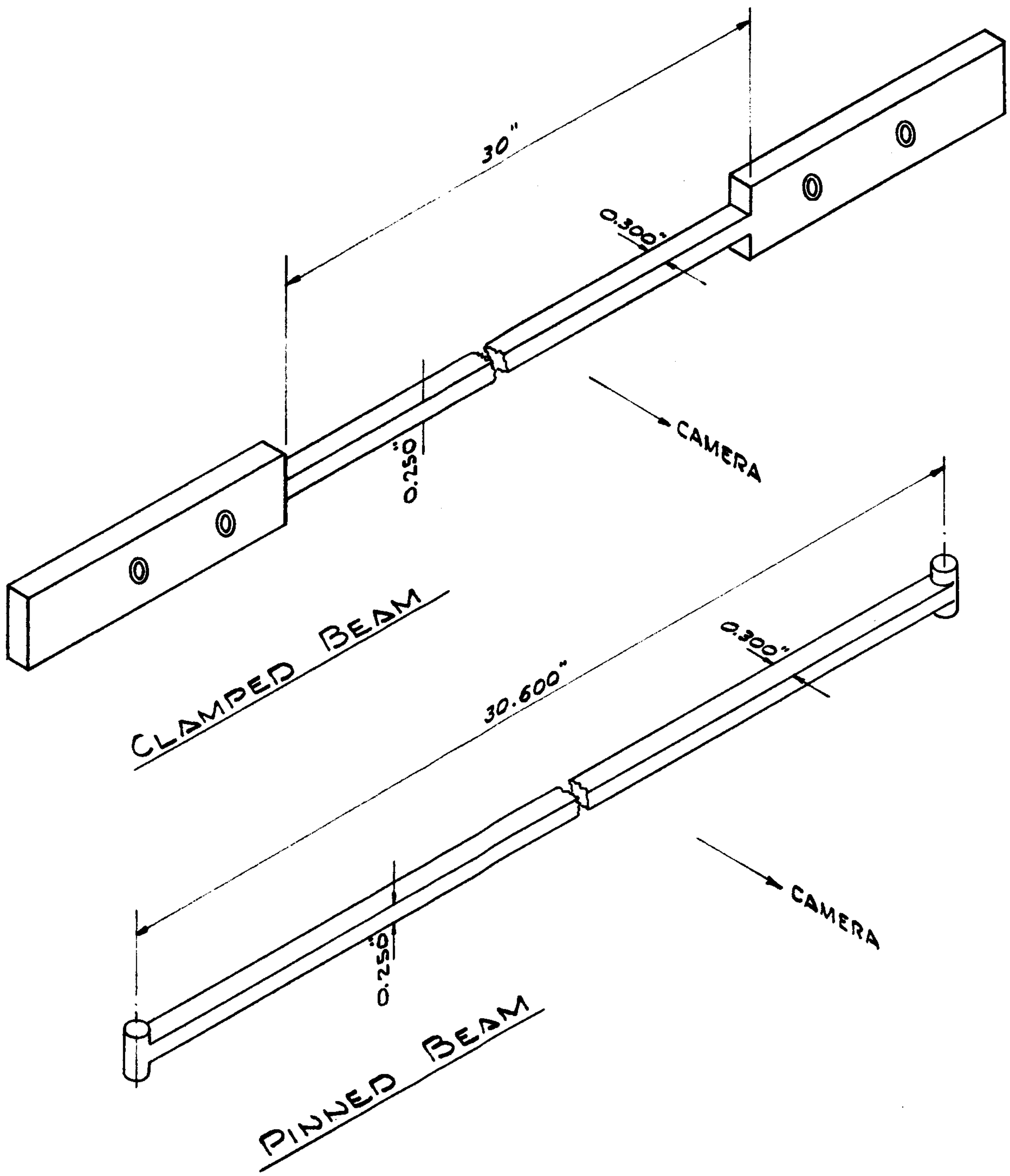

30"
0.300"
0.250"
CAMERA
CLAMPED BEAM
30.600"
0.300"
0.250"
CAMERA
PINNED BEAM

FIGURE 2

A Fastex, 16 millimeter, high speed motion camera, capable of taking 8000 pictures per second, was used together with Wollensak "Goose" Control Unit. The "Goose" control unit provides a convenient means for synchronizing, in proper time relationships, the Fastex camera operation and the event being photographed. It also provides a means for safely increasing the voltage over that normally applied to the camera so that increased camera speeds may be obtained.

For Test No. 1, the camera speed was set for an average of 5000 pictures per second. Due to focal length limitations of lenses locally available, a distance of 60 inches between grid and cantilevered beam was required to photograph the entire length of beam. A wide angle lens will be obtained to enable reduction to the design distance of 40 inches or less. After the initial exposure was taken of the beam in its neutral position, the beam was displaced by using an electromagnet to deflect the beam 1/2 inch and then suddenly released. The second exposure on the same film was made while the beam was vibrating.

Since the method of obtaining Moiré fringes necessitates rewinding of the film so that a double exposure of the event can be made, a slight change in the position of the camera resulted in a relative displacement of the images of the beam in the two exposures.

Subsequent tests with a still camera proved that the double image of the beam could be eliminated by substituting a wider surface mirror in place of the beam on the initial exposure, thereby simulating the grid lines on the beam in its neutral position. The second exposure is then taken of the vibrating beam so that only one picture of the beam itself is taken.

Test No. 2 was conducted on a portion of the cantilevered beam on one roll of film using the design distance of 40 inches between grid and specimen and a camera speed of 2000 pictures per second. Another roll of film was used to photograph the beam at a distance of 28.5 inches with a camera speed of 1000 frames per second. The grid radius relationship, 3.5d (d = distance between grid and specimen) was maintained on both sequences. Additional flood lamps were installed to increase illumination of the grid, and the decreased camera speed also allowed more light to penetrate through the aperture plate openings to the film plane.

Viewing of the developed films revealed expected Moiré patterns. Quantitative data could not be derived from these films since only a portion of the beam was photographed in these experiments, and definition was limited due to overexposure. However, we are confident that with a minimal amount of additional testing to perfect the camera settings and test procedure, quantitative results will be achieved in the near future.

LIST OF REFERENCES

1. R. M. Rosenberg, "The Normal Modes of Nonlinear N-Degree of -Freedom Systems," Journal of Applied Mechanics, Vol 29, Trans. ASME, Vol 84, Series E, March 1962, pp 7-14.

2. Thein Wah, "The Normal Modes of Vibration of Certain Nonlinear Continuous Systems," Journal of Applied Mechanics, ASME, March 1964, p 139.

3. Thein Wah, "Large Amplitude Flexural Vibration of Rectangular Plates," International Journal of Mechanical Sciences, December 1963.